STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

MR 112 /2

A lower bound for the order of a partial transversal in a Latin square.

(Journal of combinatorial theory, 7(1969), p 94-95.)

by

K.K. Koksma

1969

JOURNAL OF COMBINATORIAL THEORY 7, 94–95 (1969)

Notes

A Lower Bound for the Order of a Partial Transversal in a Latin Square

KLAAS K. KOKSMA

Mathematical Centre, Amsterdam, The Netherlands

Communicated by H. J. Ryser

Received November 11, 1968

ABSTRACT. The notion of partial transversal in a Latin square is defined. A proof is given of the existence of a partial transversal of order $> \frac{2}{3}N + \frac{1}{3}$ of a Latin square of order N ($N > 7$).

1. INTRODUCTION

A *Latin square* of order N is a square matrix, each of whose rows and columns is a permutation of the N symbols $1, 2, \ldots, N$.

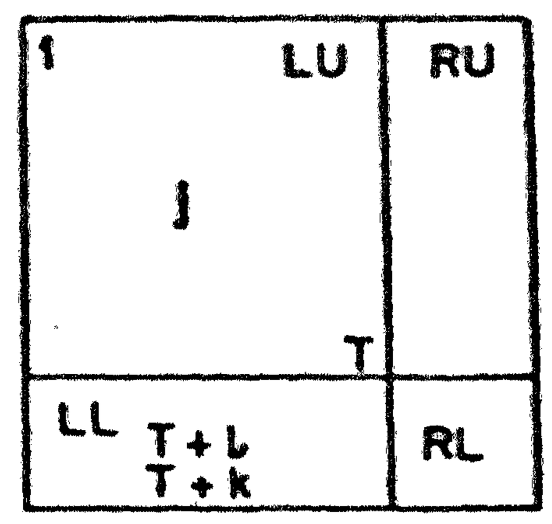
A *transversal* of a Latin square of order N is a set of N different elements of the matrix with precisely one element in every row and column. A *partial transversal* of order k of a Latin square of order N ($N \geqslant k$) is a set of k different elements of the Latin square with at most one element in every row and column.

For odd N, Ryser [1] conjectured that every Latin square has a transversal. As far as the author knows this conjecture is still undecided. So if we cannot prove the existence of a transversal, we can raise the question: How large may the order of a partial transversal be?

2. FORMULATION AND PROOF OF THE THEOREM

THEOREM. *A Latin square of order $N \geqslant 7$ has at least one partial transversal of order $\geqslant \frac{2}{3}N + \frac{1}{3}$.*

PROOF: Let S be a Latin square of order N. Let $T = T_S$ be an integer for which the following holds: (i) There is no partial transversal of S of order $> T$. (ii) There is at least one partial transversal of S of order T. Without loss of generality, the Latin square S can be divided into submatrices LU, RU, LL, and RL as indicated in the figure, and assume that the main diagonal of LU is a maximal partial transversal with elements $1, 2, \ldots, T$.



Condition (i) implies all elements in RL are $\leqslant T$; therefore, each row in LL must contain the elements $T + 1, \ldots, N$.

The total numbers of elements $> T$ in LL is $(N - T)^2$. Suppose $T + T^{1/2} < N$; then $N - T > T^{1/2}$ and $(N - T)^2 > T$. So, in this case, there must be some column, say the j-th column, which contains two elements $> T$ in LL. Because of the maximality of T we now have:

(1) j cannot occur in RL, and so occurs $N - T$ times in RU;

(2) if j occurs in RU in row $p (p \leqslant T)$, then the p-th element of the j-th row is $\leqslant T$; and

(3) all elements of the j-th row in RU are $\leqslant T$.

Combining (1), (2), and (3) and using the fact that all elements of a row of a Latin square are different, it follows that the relation $2(N - T) + 1 \leqslant T$ necessarily holds; thus,

$$T \geqslant \tfrac{2}{3}N + \tfrac{1}{3}.$$

On the other hand, $T + T^{1/2} \geqslant N \geqslant 7$ implies (using the fact T must be an integer in the case $N = 7$ and $N = 8$) immediately

$$T \geqslant \tfrac{2}{3}N + \tfrac{1}{3}$$

whence the theorem follows.

REMARK. The author has convinced himself (using trivial arguments) that the theorem also holds for N in the range $3 \leqslant N \leqslant 6$; we have omitted this proof to keep this note short.

REFERENCE

1. H. J. RYSER, Neuere Probleme der Kombinatorik im Vorträge über Kombinatorik, OBERWOLFACH 24–29 juli 1967. Mathematischen Forschungsinstitut, OBERWOLFACH, 1968.